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COORDINATES IN TWO VARIABLES OVER A $\mathbb{Q}$-ALGEBRA

ARNO VAN DEN ESSEN AND PETER VAN ROSSUM

ABSTRACT. This paper studies coordinates in two variables over a $\mathbb{Q}$-algebra. It gives several ways to characterize such coordinates. Also, various results about coordinates in two variables that were previously only known for fields, are extended to arbitrary $\mathbb{Q}$-algebras.

1. INTRODUCTION

Let R be a commutative ring. A polynomial $f \in R[X] := R[X_1, \ldots, X_n]$ is called a *coordinate in* $R[X]$ if there exists a $\varphi \in \mathrm{Aut}_R(R[X])$ such that $\varphi(X_1) = f$. The problem how to recognize coordinates is of fundamental importance in the study of polynomial automorphisms.

In the case that $n = 2$ and R is a field, this problem was solved by various authors ([CK92], [Ess93], [CMW95]). One of the deepest results here is the Abhyankar-Moh Theorem ([AM75]), which can be formulated as follows: if k is a field of characteristic zero and $f \in k[x, y]$ is such that $k[x, y]/(f) \cong_k k^{[1]}$, a polynomial ring over k in one variable, then f is a coordinate in $k[x, y]$.

Studying coordinates over rings which are not fields is much more complicated. For example, it is not immediately obvious that $y + z^2x + zy^2$ is a coordinate in $R[x, y]$, where $R = k[z]$. In fact, this polynomial is the second component of the well-known Nagata automorphism ([Nag72], [Ess00]). Such coordinates (and their automorphisms) were first investigated by Nagata ([Nag72]) and later by Drensky and Yu ([DY01]) and Edo and Vénéreau ([EV01]). The latter paper clearly demonstrates the importance of studying coordinates in two variables over commutative rings by showing how it leads to many interesting coordinates in dimensions greater than two. Even more recently, extending the results of [EV01], a large class of coordinates in two variables over commutative rings was studied by Berson in [Ber02].

This paper gives several criteria (Theorems 3.2, 3.4, and 3.5) to decide if a given polynomial in two variables over a commutative $\mathbb{Q}$-algebra is a coordinate. The first of these criteria, Theorem 3.2, is the starting point of this paper. It provides the connection between coordinates in two variables and locally nilpotent derivations of divergence zero. Using this fundamental result, various results concerning coordinates in two variables over arbitrary $\mathbb{Q}$-algebras are deduced.

For example, Section 3 shows that being a coordinate in two variables is a *residual property* (Theorem 3.4), i.e., a polynomial $f \in R[x, y]$ is a coordinate if and only

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if it is a coordinate when regarded as an element of $k_{\mathfrak{p}}[x,y]$, for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Here $k_{\mathfrak{p}}$ denotes the residue field of R in $\mathfrak{p}$. Such a result was obtained earlier by Bhatwadekar and Dutta ([BD93]), but only for the case that R is Noetherian. Section 3 also generalizes the Abhyankar-Moh Theorem mentioned earlier to arbitrary $\mathbb{Q}$-algebras (Theorem 3.5).

Section 4 studies the question if being a coordinate is a stable notion, i.e., if a polynomial $f \in R[X_1, \dots, X_n]$ is a coordinate in $R[X_1, \dots, X_n, Y_1, \dots, Y_m]$ for some $m \in \mathbb{N}$, does that imply that f is a coordinate in $R[X_1, \dots, X_n]$?

Another result concerning coordinates in $R[x,y]$ is obtained in Section 5: for an arbitrary $\mathbb{Q}$-algebra R, an R-endomorphism of $R[x,y]$ which sends every (elementary) linear coordinate to a coordinate of $R[x,y]$, is in fact an automorphism. This generalizes a result from Cheng and the first author in [CE00], where the case that R is a field of characteristic zero is treated.

In Section 6, the results on coordinates in two variables are used to investigate to what extent the famous Abhyankar-Moh-Suzuki Embedding Theorem ([AM75], [Suz74], see also [Ess00], Theorem 5.3.5) can be extended to the case where the field of coefficients is replaced by a commutative $\mathbb{Q}$-algebra.

Finally, Section 7 takes a look at polynomial rings in more than two variables. It gives a partial generalization of a theorem of Sathaye and Russel ([Sat76], [Rus76]) concerning the Abhyankar-Sathaye Conjecture. Furthermore, it gives several concrete examples showing that the results from the previous section cannot generally be extended to polynomial rings in more than two variables.

2. Preliminaries

Throughout this paper, R denotes a commutative ring and $R[X] := R[X_1, \dots, X_n]$ denotes the polynomial ring in n variables over R. This ring is also written as $R^{[n]}$. If $\mathfrak{p}$ is a prime ideal of R, then the residue field of R at $\mathfrak{p}$ is denoted by $k_{\mathfrak{p}}$. If $f \in R[X]$, then the corresponding element of $k_{\mathfrak{p}}[X]$ is written as $f_{\mathfrak{p}}$. Finally, if D is an R-derivation on $R[X]$, it naturally gives rise to a $k_{\mathfrak{p}}$-derivation on $k_{\mathfrak{p}}[X]$. This induced derivation is denoted by $D_{\mathfrak{p}}$.

In order to prepare for the proofs of the main theorems of this paper, this section collects some more or less well-known facts about derivations on polynomial rings. The first result (Proposition 2.2) asserts that being locally nilpotent is a residual property of an R-derivation on $R[X]$. The following lemma, which is essentially Lemma 2.1.15 of [Ess00], is needed in the proof.

Lemma 2.1. *Let D be an R-derivation on $R[X]$. Let η be the nilradical of R and denote by D/η the derivation on $R/\eta[X]$ induced by D. Assume that D/η is locally nilpotent. Then D is locally nilpotent as well.*

Proof. Let R' be the subring of R generated by all coefficients appearing in the polynomials $D(X_1), \dots, D(X_n)$. Then R' is Noetherian and D restricts to a derivation D' on $R'[X]$. Note that D is locally nilpotent if and only if D' is. Also note that the nilradical η' of R' equals $\eta \cap R'$ and that hence D/η is locally nilpotent if and only if D'/η' is. Therefore it is possible to assume, without loss of generality, that the ring R is Noetherian.

By induction on m it will follow that for every $m \in \mathbb{N}^*$ and every $g \in R[X]$, there exists an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^m[X]$. For $m = 1$, this is just the assumption that D/η is locally nilpotent. Now assume that the claim holds for m and consider

$g \in R[X]$. By the induction hypothesis, $D^N(g) \in \eta^m[X]$ for some $N \in \mathbb{N}$, say $D^N(g) = \sum_{\alpha \in A} c_\alpha X^\alpha$, for certain $c_\alpha \in \eta^m$. Since D/η is locally nilpotent, there are $M_\alpha \in \mathbb{N}$ such that $D^{M_\alpha}(X^\alpha) \in \eta[X]$. Taking $M := N + \max_{\alpha \in A} M_\alpha$, it follows that $D^M(g) \in \eta^{m+1}[X]$.

Because R is Noetherian, its nilradical η is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Consequently, for every $g \in R[X]$ there is an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^e[X] = (0)$. Therefore D is locally nilpotent. □

Proposition 2.2. *Assume that R is Noetherian and let D be an R-derivation on $R[X]$. Then the following two statements are equivalent:*

(1) *D is locally nilpotent;*
(2) *for every $\mathfrak{p} \in \operatorname{Spec}(R)$, the $k_\mathfrak{p}$-derivation $D_\mathfrak{p}$ on $k_\mathfrak{p}[X]$ is locally nilpotent.*

Proof. The implication (1) ⇒ (2) is clear. So assume that for every $\mathfrak{p} \in \operatorname{Spec}(R)$, the derivation $D_\mathfrak{p}$ is locally nilpotent.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Consider the ring $R/\mathfrak{p}$ and the derivation $D/\mathfrak{p}$ induced by D on $R/\mathfrak{p}[X]$. Because $Q(R/\mathfrak{p}) = k_\mathfrak{p}$ and the derivation $D_\mathfrak{p}$ on $k_\mathfrak{p}[X]$ is locally nilpotent, $D/\mathfrak{p}$ is locally nilpotent as well.

So $D/\mathfrak{p}$ is locally nilpotent for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Since R is Noetherian, its nilradical η is a finite intersection of prime ideals, say $\eta = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$. Because $D/\mathfrak{p}_i$ is locally nilpotent for every $i \in \{1, \ldots, s\}$, D/η is locally nilpotent too. Namely, let $g \in R[X]$. Then there is an $N_i \in \mathbb{N}$ such that $D^{N_i}(g) \in \mathfrak{p}_i$, for every $i \in \{1, \ldots, s\}$. Taking $N := \max_{i \in \{1,\ldots,s\}} N_i$ it follows that $D^N(g) \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s = \eta$. By Lemma 2.1, D is locally nilpotent. □

In dimension two, the condition that the ring R is Noetherian can be avoided. In order to prove this, the following result is needed ([Ess00], Theorem 1.3.49).

Lemma 2.3. *Let k be a field of characteristic 0 and take $D \in \operatorname{Der}_k(k[x,y])$. Assume that $D \neq 0$ and let*

$$d := \max\{\deg_x D(x), \deg_x D(y), \deg_y D(x), \deg_y D(y)\}.$$

(Here, by convention, the degree of 0 is taken to be $-\infty$.) Then D is locally nilpotent if and only if $D^{d+2}(x) = D^{d+2}(y) = 0$. □

Proposition 2.4. *Assume that R is a $\mathbb{Q}$-algebra and let D be an R-derivation on $R[X]$. Then the following two statements are equivalent:*

(1) *D is locally nilpotent;*
(2) *for every $\mathfrak{p} \in \operatorname{Spec}(R)$, the $k_\mathfrak{p}$-derivation $D_\mathfrak{p}$ on $k_\mathfrak{p}[x,y]$ is locally nilpotent.*

Proof. The implication (1) ⇒ (2) is once again clear, so assume that for every $\mathfrak{p} \in \operatorname{Spec}(R)$, the derivation $D_\mathfrak{p}$ is locally nilpotent. Take $d := \max\{\deg_x D(x), \deg_x D(y), \deg_y D(x), \deg_y D(y)\}$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Just as in the proof of Proposition 2.2, consider the ring $R/\mathfrak{p}$ and the locally nilpotent derivation $D/\mathfrak{p}$ induced by D on $R/\mathfrak{p}[X,Y]$ and on $Q(R/\mathfrak{p})[X,Y]$. Note that $Q(R/\mathfrak{p})$ is a field of characteristic 0, since R is a $\mathbb{Q}$-algebra. Hence, by the previous lemma,

$$(D/\mathfrak{p})^{d+2}(x) = (D/\mathfrak{p})^{d+2}(y) = 0,$$

or, differently said, $D^{d+2}(x) \in \mathfrak{p}[x,y]$ and $D^{d+2}(y) \in \mathfrak{p}[x,y]$.

Hence $D^{d+2}(x) \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathfrak{p}[x,y] = \eta[x,y]$, where η is the nilradical of R, and similarly $D^{d+2}(y) \in \eta[x,y]$. This means that D/η is locally nilpotent and hence, by Lemma 2.1, D is locally nilpotent too. $\square$

Another result which will play a crucial role in the next section is the following theorem. It is Theorem 3.7 of [BEM01].

Theorem 2.5. *Assume that R is a $\mathbb{Q}$-algebra and let D be a locally nilpotent R-derivation on $R[x,y]$. Assume that D is of the form $f_y\partial_x - f_x\partial_y$ for some $f \in R[x,y]$. If $(f_x, f_y) = (1)$, then D has a slice in $R[x,y]$ and $R[x,y]^D = R[f]$.* $\square$

The following two residual properties will also be used in the next section.

Lemma 2.6. *Let $\mathfrak{a}$ be an ideal of $R[X]$. Then the following two statements are equivalent:*

(1) $\mathfrak{a} = R[X]$;
(2) *for every* $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{a}_\mathfrak{p} = k_\mathfrak{p}[X]$.

Here $\mathfrak{a}_\mathfrak{p}$ denotes the ideal of $k_\mathfrak{p}[X]$ generated by the polynomials $f_\mathfrak{p}$, $f \in \mathfrak{a}$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. So assume that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{a}_\mathfrak{p} = k_\mathfrak{p}[X]$.

Assume that $\mathfrak{a} \neq R[X]$. Then there is some maximal ideal $\mathfrak{m}$ of $R[X]$ such that $\mathfrak{a} \subseteq \mathfrak{m}$. Let $\mathfrak{p} := \mathfrak{m} \cap R$. This is a prime ideal of R. Using the natural isomorphism between $k_\mathfrak{p}$ and $Q(R/\mathfrak{p})$, one can easily see that $1 \in \mathfrak{a}_\mathfrak{p}$ means that there are an $r \in R \setminus \mathfrak{p}$, polynomials $g_1, \ldots, g_s \in R[X]$, and polynomials $f_1, \ldots, f_s \in \mathfrak{a}$ such that

$$r \equiv g_1 f_1 + \cdots + g_s f_s \pmod{\mathfrak{p}[X]}.$$

Because $\mathfrak{a} \subseteq \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$, this implies, however, that $r \in \mathfrak{m}$. Because also $r \in R$, this contradicts the fact that $r \notin \mathfrak{p} = m \cap R$. Therefore $\mathfrak{a} = R[X]$. $\square$

Proposition 2.7. *Let $F \in \mathrm{End}_R(R[X])$. Then the following two statements are equivalent:*

(1) $F \in \mathrm{Aut}_R(R[X])$;
(2) *for every* $\mathfrak{p} \in \mathrm{Spec}(R)$, $F_\mathfrak{p} \in \mathrm{Aut}_{k_\mathfrak{p}}(k_\mathfrak{p}[X])$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. So assume that $F_\mathfrak{p}$ is an automorphism of $k_\mathfrak{p}[X]$ for all prime ideals $\mathfrak{p}$ of R. Let $d := \det(JF)$. Because each $F_\mathfrak{p}$ is an automorphism, $d_\mathfrak{p} \in k_\mathfrak{p}^*$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Therefore, by Lemma 2.6, $d \in R[X]^*$. Note that it is possible to assume, without loss of generality, that $F_i(0) = 0$ for all $i \in \{1, \ldots, n\}$. Then by the formal inverse function theorem ([Ess00], Theorem 1.1.2) F has a uniquely determined formal inverse $G = (g_1, \ldots, g_n)$ with each $g_i \in R[[X]]$.

Take $N := (\deg F)^{n-1}$. Since each $F_\mathfrak{p}$ is invertible, it follows from Proposition 2.3.1 of [Ess00] that $\deg G_\mathfrak{p} \leq N$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. This means that if $c_{i,\alpha}$ is the coefficient of the monomial X^α in g_i, then $c_{i,\alpha} = 0$ in $k_\mathfrak{p}$ if $|\alpha| > N$. It follows that $c_{i,\alpha} \in \eta$, the nilradical of R, if $|\alpha| > N$.

Write $\bar{R} := R/\eta$ and let $\bar{F}$ be the $\bar{R}$-endomorphism of $\bar{R}[X]$ obtained by reducing the coefficients of F modulo η. Then $\bar{F}$ is invertible with inverse $\bar{G}$. Hence, by Lemma 1.1.3 of [Ess00], F is invertible. $\square$

3. Criteria for coordinates in two variables

Recall that a polynomial $f \in R[X]$ is called a *coordinate in* $R[X]$ if there exists a $\varphi \in \mathrm{Aut}_R(R[X])$ such that $\varphi(X_1) = f$, or, equivalently, if there are $f_2, \dots, f_n \in R[X]$ such that $R[X] = R[f, f_2, \dots, f_n]$. We will also say that f is a *coordinate over* R. In this case, the determinant of the Jacobian matrix $J(f, f_2, \dots, f_n)$ is a unit in $R[X]$. Taking the Laplace expansion of this determinant according to its first row, it follows that 1 belongs to the ideal generated by the partial derivatives of f. This motivates the following definition.

Definition 3.1. A polynomial $f \in R[X]$ is called *unimodular* if $(f_{X_1}, \dots, f_{X_n}) = R[X]$.

So the argument given above shows that every coordinate of $R[X]$ is unimodular. Conversely, if f is unimodular in $R[X]$, then already in the case that $n = 2$ the polynomial f does not have to be a coordinate in $R[X]$. For example, take the unimodular polynomial $f := X_1 + X_1^2 X_2$ in $R[X_1, X_2]$. This is not a coordinate, because it is reducible. The following theorem, in fact, describes exactly which additional condition is necessary and sufficient to guarantee that a polynomial in two variables over a $\mathbb{Q}$-algeba is a coordinate.

Theorem 3.2. *Let $f \in R[x, y]$, and let D be the R-derivation $f_y \partial_x - f_x \partial_y$ on $R[x, y]$. Then the following three statements are equivalent:*

(1) *D is locally nilpotent and f is unimodular;*
(2) *D is locally nilpotent, has a slice, and $R[x, y]^D = R[f]$;*
(3) *f is a coordinate.*

Proof. The equivalence of (1) and (2) is Theorem 2.5. The implication (2) $\Rightarrow$ (3) follows immediately from Proposition 2.1 of [Wri81] (see also Proposition 1.3.21 of [Ess00]). So the only thing left to prove is the implication (3) $\Rightarrow$ (2).

Let $f \in R[x, y]$ be a coordinate. Then there exists a polynomial $g \in R[x, y]$ such that (f, g) is an invertible polynomial map over R. Let η denote the nilradical of R. To avoid notational clutter, reduction modulo this nilradical η of R or modulo the nilradical $\eta[x, y]$ of $R[x, y]$ will be denoted by an overline.

Then $(\bar{f}, \bar{g})$ is an invertible polynomial map over $\bar{R}$ and hence $\bar{R}[x, y] = \bar{R}[\bar{f}, \bar{g}]$. Now note that

$$\bar{D}(\bar{g}) = \det J(\bar{f}, \bar{g}) \in \bar{R}[x, y]^* = \bar{R}^*$$

and so $\bar{D}^2(\bar{g}) = 0$. Also $\bar{D}(\bar{f}) = 0$ and therefore $\bar{D}$ is locally nilpotent. By Lemma 2.1, D is locally nilpotent too. Also, as observed above, f is unimodular. Applying Theorem 2.5 gives (2). □

If R is not a $\mathbb{Q}$-algebra, then the implication (1) $\Rightarrow$ (3) does not hold: take $R := \mathbb{Z}[t]/(2t)$ and $f := x - \bar{t}x^2$. Then $(f_x, f_y) = (1)$ and the derivation $f_y \partial_x - f_x \partial_y = -\partial_y$ is locally nilpotent. Furthermore, f is not a coordinate in $R[x]$ (see [Ess00], 1.1.17). From the next lemma it then follows that f is not a coordinate in $R[x, y]$ either.

Lemma 3.3. *Let $f_1(X), \dots, f_n(X) \in R[X] := R[X_1, \dots, X_n]$ and write $Y := Y_1, \dots, Y_m$. If there are $g_1(X, Y), \dots, g_m(X, Y) \in R[X, Y]$ such that $R[f_1, \dots, f_n, g_1, \dots, g_m] = R[X, Y]$, then $R[f_1, \dots, f_n] = R[X]$.*

Proof. Let $g_1(X, Y), \dots, g_m(X, Y) \in R[X, Y]$ be polynomials such that

$$R[f_1, \dots, f_n, g_1, \dots, g_m] = R[X, Y].$$

This means that (f, g) is an R-automorphism of $R[X, Y]$. Let $(h_1(X, Y), \ldots, h_{m+n}(X, Y))$ be its inverse. Then, in particular,

$$X_i = f_i(h_1(X, Y), \ldots, h_n(X, Y))$$

for all $i \in \{1, \ldots, n\}$. Substituting $Y_j := 0$ for all $j \in \{1, \ldots, m\}$ shows that $R[f_1, \ldots, f_n] = R[X]$. □

Using Theorem 3.2 it is now possible to show that being a coordinate in $R[x, y]$ is a residual property.

Theorem 3.4. *Assume that R is a $\mathbb{Q}$-algebra and let $f \in R[x, y]$. Then the following two statements are equivalent:*

(1) *f is a coordinate over R in $R[x, y]$;*
(2) *for every $\mathfrak{p} \in \operatorname{Spec}(R)$, $f_{\mathfrak{p}}$ is a coordinate in $k_{\mathfrak{p}}[x, y]$.*

Proof. The implication (1) ⇒ (2) is obvious, so assume that $f_{\mathfrak{p}}$ is a coordinate in $k_{\mathfrak{p}}[x, y]$ for every prime ideal $\mathfrak{p}$ of R. Write $D := f_y\partial_x - f_x\partial_y$. From Theorem 3.2 it follows that $D_{\mathfrak{p}}$ is locally nilpotent on $k_{\mathfrak{p}}[x, y]$ and that $(f_x, f_y) = k_{\mathfrak{p}}[x, y]$. Consequently it follows from Proposition 2.4 and Lemma 2.6 that D is locally nilpotent on $R[x, y]$ and that f is unimodular in $R[x, y]$. So, using Theorem 3.2, f is a coordinate in $R[x, y]$. □

The condition that R is a $\mathbb{Q}$-algebra cannot simply be dropped in this theorem. For Bhatwadekar and Dutta have constructed the following example in [BD93]. Take $R := \mathbb{Z}_{2\mathbb{Z}}[2\sqrt{2}]$ and take

$$f := x - 2y(\sqrt{2}x - y^2) + \sqrt{2}(\sqrt{2}x - y^2)^2 - \sqrt{2}(y - \sqrt{2}(\sqrt{2}x - y^2))^4.$$

Then $f_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}$, for every prime ideal $\mathfrak{p}$ of R, but f itself is not a coordinate over R.

Finally, as a third criterion for recognizing coordinates in two variables, the Abhyankar-Moh Theorem is generalized to arbitarary $\mathbb{Q}$-algebras.

Theorem 3.5. *Assume that R is $\mathbb{Q}$-algebra and let $f \in R[x, y]$. Assume that $R[x, y]/(f) \cong R^{[1]}$. Then f is a coordinate in $R[x, y]$.*

Proof. Let D be the derivation $f_y\partial_x - f_x\partial_y$ on $R[x, y]$.

Consider a prime ideal $\mathfrak{p}$ of R. Then $k_{\mathfrak{p}}[X, Y]/(f_{\mathfrak{p}}) \cong k_{\mathfrak{p}}^{[1]}$. Since R is a $\mathbb{Q}$-algebra, $k_{\mathfrak{p}}$ is a field of characteristic 0. So the Abhyankar-Moh Theorem implies that $f_{\mathfrak{p}}$ is a coordinate over $k_{\mathfrak{p}}$ in $k_{\mathfrak{p}}[x, y]$.

Now Theorem 3.4 implies that f itself is a coordinate over R. □

4. Coordinates in two variables under ring extensions

The main result of this section, Theorem 4.2, gives a criterion which decides if a polynomial in $R[X_1, X_2]$ which is a coordinate in a larger polynomial ring $S[X_1, \ldots, X_n]$ for some $n \geq 2$ and some ring extension $R \subseteq S$, is a coordinate in $R[X_1, X_2]$.

Lemma 4.1. *Let $K \subseteq L$ be a field extension and let $\mathfrak{a}$ be an ideal of the polynomial ring $K[X_1, \ldots, X_n]$. Let $\mathfrak{b}$ be the ideal of $L[X_1, \ldots, X_n]$ generated by the elements of $\mathfrak{a}$. If $1 \in \mathfrak{b}$, then $1 \in \mathfrak{a}$.*

Proof. Left to the reader. □

Theorem 4.2. *Let $R \subseteq S$ be an extension of $\mathbb{Q}$-algebras, $f \in R[X_1, X_2]$, and let $n \geq 2$. Assume that (at least) one of the following two conditions holds:*

(a) *f is unimodular;*
(b) *$R \subseteq S$ satisfies the going-up property, i.e., for every $\mathfrak{p} \in \operatorname{Spec}(R)$ there is a $\mathfrak{q} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} = \mathfrak{q} \cap R$.*

Then the following four statements are equivalent:

(1) *f is a coordinate in $R[X_1, X_2]$;*
(2) *f is a coordinate in $R[X_1, \ldots, X_n]$;*
(3) *f is a coordinate in $S[X_1, \ldots, X_n]$;*
(4) *$S[X_1, \ldots, X_n]/(f) \cong_S S^{[n-1]}$.*

Proof. It is enough to show (4) $\Rightarrow$ (1). Let $\mathfrak{q} \in \operatorname{Spec}(S)$. Then

$$k_{\mathfrak{q}}[X_1, \ldots, X_n]/(f_{\mathfrak{q}}) \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[n-1]},$$

so $k_{\mathfrak{q}}[X_1, X_2]/(f_{\mathfrak{q}})[X_3, \ldots, X_n] \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[n-1]}$. It now follows from Corollary 2.8 of [AHE72] that $k_{\mathfrak{q}}[X_1, X_2]/(f_{\mathfrak{q}}) \cong_{k_{\mathfrak{q}}} k_{\mathfrak{q}}^{[1]}$ and hence the Abhyankar-Moh Theorem implies that $f_{\mathfrak{q}}$ is a coordinate in $k_{\mathfrak{q}}[X_1, X_2]$.

So, by Theorem 3.4, f is a coordinate in $S[X_1, X_2]$. Consequently, the derivation $D := f_{X_2}\partial_{X_1} - f_{X_1}\partial_{X_2}$ is locally nilpotent on $S[X_1, X_2]$ and hence on $R[X_1, X_2]$.

(a) Firstly, assume that condition (a) is satisfied. Because f is unimodular, Theorem 3.2 now implies that f is a coordinate in $R[X_1, X_2]$.
(b) Secondly, assume that condition (b) is satisfied. It is enough to show that $(f_{\mathfrak{p}X_1}, f_{\mathfrak{p}X_2}) = k_{\mathfrak{p}}[X_1, X_2]$ for all prime ideals $\mathfrak{p}$ of R, because then Lemma 2.6 implies that f is unimodular in $R[X_1, X_2]$ and Theorem 3.2 can be applied.

   So let $\mathfrak{p}$ be a prime ideal of R and choose a prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap R = \mathfrak{p}$. So $R/\mathfrak{p} \subseteq S/\mathfrak{q}$ and therefore $k_{\mathfrak{p}} \subseteq k_{\mathfrak{q}}$. Furthermore $f_{\mathfrak{q}}$ is a coordinate in $k_{\mathfrak{q}}[X_1, X_2]$ and therefore $(f_{\mathfrak{q}X_1}, f_{\mathfrak{q}X_2}) = k_{\mathfrak{q}}[X_1, X_2]$. Applying Lemma 4.1 to the field extension $k_{\mathfrak{p}} \subseteq k_{\mathfrak{q}}$ gives $(f_{\mathfrak{p}X_1}, f_{\mathfrak{p}X_2}) = k_{\mathfrak{p}}[X_1, X_2]$. □

Taking $S = R$ in this theorem gives the following generalization of Theorem 3.5.

Corollary 4.3. *Assume that R is a $\mathbb{Q}$-algebra. Let $f \in R[X_1, X_2]$ and $n \geq 2$. Then f is a coordinate in $R[X_1, X_2]$ if and only if f is a coordinate in $R[X_1, \ldots, X_n]$ if and only if $R[X_1, \ldots, X_n]/(f) \cong_R R^{[n-1]}$.* □

Also note that if neither of the conditions (a) and (b) is satisfied, then the implication (4) $\Rightarrow$ (1) of Theorem 4.2 is obviously false: take $R := \mathbb{Q}[t]$, $S := \mathbb{Q}[t, t^{-1}]$, and $f := tX_1$.

One might think that the condition $S^* \cap R = R^*$ is the only obstruction to the implication (4) $\Rightarrow$ (1). However, the example $R := \mathbb{Q}[a, b]$, $S := \mathbb{Q}[a, b, c, d]/(ab - bc - 1)$, $f := aX_1 + bX_2$ shows that this is not the case. Here $S^* \cap R = R^*$, as one easily verifies, and f is a coordinate in $S[x, y]$, but not in $R[x, y]$ (since f is not unimodular in $R[x, y]$).

5. Endomorphisms sending linear coordinates to coordinates

Assume that R is a $\mathbb{Q}$-algebra. The main result of [CE00] asserts that if R is a field, then every R-endomorphism of $R[x, y]$ sending all linear coordinates to coordinates is an R-automorphism of $R[x, y]$. This section generalizes this result to arbitrary $\mathbb{Q}$-algebras.

Definition 5.1. A linear polynomial $ax + by \in R[x, y]$ is called an *elementary linear coordinate* if $a = 1$ or $b = 1$.

Theorem 5.2. *Let φ be an R-endomorphism of $R[x, y]$ and assume that φ sends every elementary linear coordinate to a coordinate in $R[x, y]$. Then φ is an R-automorphism of $R[x, y]$.*

Proof. Since $\varphi(x)$ is a coordinate of $R[x, y]$, there exists an R-automorphism ψ of $R[x, y]$ with $\psi(x) = \varphi(x)$. So $\psi^{-1}\varphi$ is an R-endomorphism of $R[x, y]$ which sends every elementary linear coordinate of $R[x, y]$ to a coordinate of $R[x, y]$ and which sends x to x. It is obviously enough to show that $\psi^{-1}\varphi$ is an R-automorphism and hence it is possible to assume, replacing φ by $\psi^{-1}\varphi$, that φ is of the form (x, g) for some $g \in R[x, y]$.

By Proposition 2.7, it suffices to show that $(x, g_\mathfrak{p})$ is a $k_\mathfrak{p}$-automorphism of $k_\mathfrak{p}[x, y]$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

So let $\mathfrak{p}$ be a fixed prime ideal of R. Since $g - cx = \varphi(y - cx)$ is a coordinate for all $c \in R$, it follows that $g - \bar{c}x$ is a coordinate in $k_\mathfrak{p}[x, y]$ for all $c \in R$. Here $\bar{c} := c + \mathfrak{p} \in R/\mathfrak{p} \subseteq k_\mathfrak{p}$. Hence, taking for c the coefficient of the monomial x appearing in g, it follows that $\deg_y g > 0$. Write

$$g = g_n y^n + g_{n-1} y^{n-1} + \cdots + g_0,$$

with each $g_i \in k[x]$, $n \geq 1$ and $g_n \neq 0$. Since $g = \varphi(y)$ is a coordinate in $R[x, y]$, g is also a coordinate when regarded as an element of $k_\mathfrak{p}[x, y]$. Then by [AE90], Corollary 1.4 or [Ess00], Corollary 3.3.7, it follows that $g_n \in k_\mathfrak{p}^*$. Consequenly, it is enough to show that $n = 1$, for then $\varphi_\mathfrak{p} = (x, g_1 y + g_0)$, which is obviously a $k_\mathfrak{p}$-automorphism of $k_\mathfrak{p}[x, y]$.

Assume that $n \geq 2$. Replacing g by $g_n^{-1} g$ it is possible to assume that $g_n = 1$. Furthermore, replacing y by $y - n^{-1} g_{n-1}$ it is possible to assume that $g_{n-1} = 0$. Now we follow the proof of Theorem 1.1 of [CE00].

Put $D := g_y \partial_x - g_x \partial_y$. Since $\mathbb{Q} \subseteq k_\mathfrak{p}$, it follows that $(D + q\partial_y)^n(x) \in k_\mathfrak{p}$ for all $q \in \mathbb{Q}$. Then by Lemma 1.3 of [CE00] (using the fact that $\mathbb{Q}$ is infinite) it follows that the polynomial $h(t) := (D + t\partial_y)^n(x)$ is an element of $k_\mathfrak{p}[t]$. So, in particular, the coefficient of t^{n-2} of $h(t)$ belongs to k. Then by Proposition 2.1 of [CE00] we get that $g_x \in k$. So $g = \lambda x + a(y)$ for some $\lambda \in k$ and some $a(y) \in k[y]$. Since $g \in R[x, y]$, the coefficients of g belong in fact to $\bar{R} := R/\mathfrak{p} \subseteq k$, i.e., $\lambda = \bar{c}$ for some $c \in R$. Then again using the fact that $g - \bar{c}x$ is a coordinate in $k[x, y]$, we get that $a(y)$ is a coordinate in $k[x, y]$ and hence in $k[y]$ (by Lemma 3.3). But this is a contradiction, since $\deg a(y) = n \geq 2$. □

6. Coordinates and embeddings

The famous Abhyankar-Moh-Suzuki Theorem ([AM75], [Suz74]; see also [Ess00], Theorem 5.3.5) asserts that every embedding of the affine line in the plane is rectifiable. In algebraic terms this means that if k is a field of characteristic zero and $f_1(t), f_2(t) \in k[t]$ are such that $k[t] = k[f_1, f_2]$, then there is an invertible polynomial map $F\colon k^2 \to k^2$ such that $F \circ (f_1(t), f_2(t)) = (t, 0)$. The aim of this section is to investigate to what extent this result can be generalized.

First some notations are needed. Consider a polynomial map $\alpha\colon R \to R^n$ given by $\alpha(t) = (f_1(t), \ldots, f_n(t))$ for certain polynomials $f_1, \ldots, f_n \in R[t]$. Then α is called an *embedding of R in R^n* if $R[f_1, \ldots, f_n] = R[t]$. Equivalently, α is called an embedding if the induced ring homomorphism $\alpha^*\colon R[X_1, \ldots, X_n] \to R[t]$, which sends each X_i to f_i, is surjective. Such an embedding is called *rectifiable* if there is an invertible polynomial map $F\colon R^n \to R^n$ such that $F \circ \alpha = (t, 0, \ldots, 0)$. Equivalently, α is rectifiable if $\alpha^* \circ F^* = i^*$. Here $F^*\colon R[X] \to R[X]$ is the R-automorphism of $R[X]$ induced by F and $i^*\colon R[X] \to R[t]$ is the ring homomorphism induced by the inclusion $i\colon R \to R^n$, $i(t) = (t, 0, \ldots, 0)$.

Lemma 6.1. *Let α be an embedding of R in R^n. Then the embedding $\tilde{\alpha}$ of R in R^{n+1} given by $\tilde{\alpha} = (\alpha, 0)$ is rectifiable.*

Proof. Assume that α is given by $f_1, \ldots, f_n \in R[t]$. Let $p \in R[X_1, \ldots, X_n]$ such that $p(f_1, \ldots, f_n) = t$. Take $G := (X_1, \ldots, X_n, X_{n+1} + p)$ and $H := (X_1 - f_1(X_{n+1}), \ldots, X_n - f_n(X_{n+1}), X_{n+1})$. Then G and H are invertible polynomial maps from R^{n+1} to itself. Take $F := H \circ G$. Then $F \circ \tilde{\alpha} = (0, t)$, showing that $\tilde{\alpha}$ is rectifiable. □

The following lemma shows the relation between rectifiable embeddings and coordinates.

Lemma 6.2. *Let α be an embedding of R in R^n. Then α is rectifiable if and only if* $\mathrm{Ker}(\alpha^*)$ *contains a coordinate.*

Proof. The implication $\Rightarrow$ is easy. So suppose that $\mathrm{Ker}(\alpha^*)$ contains a coordinate. Let F be an invertible polynomial map from R^n to R^n which has this coordinate as its last component. Then, using the notations of the proof of the previous lemma, $F \circ \alpha$ is an embedding of the form $(f_1(t), \ldots, f_{n-1}(t), 0)$. So, by Lemma 6.1, $F \circ \alpha$ is rectifiable. This implies that α is rectifiable. □

In dimension two it is possible to use the generalized Abhyankar-Moh Theorem of Section 3 to obtain the following.

Theorem 6.3. *Let $\alpha\colon R \to R^2$ be an embedding. Then α is rectifiable if and only if* $\mathrm{Ker}(\alpha^*)$ *is a principal ideal.*

Proof. ($\Rightarrow$) Suppose that F is an invertible polynomial map from R^2 to R^2 rectifying α. Consider $F^* \in \mathrm{Aut}_R(R[x,y])$, $\alpha^* \in \mathrm{Hom}_R(R[x,y], R[t])$, and $i^* \in \mathrm{Hom}_R(R[x,y], R[t])$, associated to F, α, and the embedding $i\colon R \to R^2$, $i(t) = (t, 0)$, respectively. Then $\alpha^* \circ F^* = i^*$. So, for all $g \in R[x,y]$, $\alpha^*(g) = 0$ iff $\alpha^* \circ F^*(F^{*-1}(g)) = 0$ iff $i^*(F^{*-1}(g)) = 0$ iff $F^{*-1}(g) \in \mathrm{Ker}(i^*) = (y)$ iff $g \in (F^*(y))$. Therefore $\mathrm{Ker}(\alpha^*) = (F^*(y))$, which implies that $\mathrm{Ker}(\alpha^*) = (F^*(y))$.

($\Leftarrow$) Conversely, suppose that $\mathrm{Ker}(\alpha^*) = (f)$ for some $f \in R[x,y]$. Then $R[x,y]/(f) \cong_R R[t]$ and hence, by Theorem 3.5, f is a coordinate. So by Lemma 6.2, α is rectifiable. □

Now we are able to give the main result of this section.

Theorem 6.4. (1) *Assume that R is a $\mathbb{Q}$-algebra and assume that R is a unique factorization domain. Then every embedding $\alpha\colon R \to R^2$ is rectifiable.*

(2) *Let $R := \mathbb{C}[z^2, z^3] \subseteq \mathbb{C}[z]$. Take $f := t - z^3t^2$ and $g := z^2t$ in $R[t]$. Then $\alpha := (f, g)$ is a non-rectifiable embedding of R in R^2.*

Proof. (1) Denote the quotient field $Q(R)$ of R by k. Note that α can also be regarded as an embedding of k in k^2.

By the Abhyankar-Moh-Suzuki Theorem, $\alpha\colon k \to k^2$ is rectifiable and therefore there exists a polynomial $f \in k[x,y]$ such that $\operatorname{Ker}(\alpha^*\colon k[x,y] \to k[t]) = k[x,y]f$. It is possible to choose f in such a way that $f \in R[x,y]$ and f is primitive (over R). Then $\operatorname{Ker}(\alpha^*\colon R[x,y] \to R[t]) = R[x,y] \cap k[x,y]f = R[x,y]f$. Now apply the previous theorem.

(2) Take $F := x + z^3x^2 - z^3xy^3 + 2y^3 - z^2y^5 \in R[x,y]$. Then one easily verifies that $F(f,g) = u$. Hence α is an embedding.

To show that α is not rectifiable, it is enough to show that $\operatorname{Ker}(\alpha^*)$ is not a principal ideal (Theorem 6.3). So assume that $\operatorname{Ker}(\alpha^*) = (p)$ for some $p \in R[x,y]$.

Take $b := z^4x - z^2y + z^3y^2 \in R[x,y]$. Then $b \in \operatorname{Ker}(\alpha^*)$, so $b = ap$ for some $a \in R[x,y]$. Looking at the coefficients of x and 1 in the x-expansions of b and ap, one easily deduces that $p = \lambda b$, for some $\lambda \in \mathbb{C}^*$. So $\operatorname{Ker}(\alpha^*) = (b)$.

Finally, $c := -y + z^2x + z^3xy + z^2y^3 \in \operatorname{Ker}(\alpha^*)$. Write $c = \tilde{a}b$ for some $\tilde{a} \in R[x,y]$. Substituting $z := 0$ and $x := 0$ in this equation gives $-1 = 0$, a contradiction. So α is not rectifiable. □

The first part of this theorem was already obtained by Berson in [Ber99].

7. Remarks on coordinates in dimensions greater than two

In the previous sections various results for coordinates in two variables were given. This section considers coordinates in more than two variables.

The Abhyankar-Sathaye Conjecture states the following: for $n \geq 3$, if k is a field of characteristic zero and $f \in k[X_1, \ldots, X_n]$ is a polynomial such that $k[X]/(f) \cong k^{[n-1]}$, then f is a coordinate. The following special case of the conjecture was proven by Sathaye ([Sat76]) and Russell ([Rus76]).

Theorem 7.1. *Let k be a field and let $f(x,y,z)$ be a polynomial over k of the form*

$$f(x,y,z) = g(x,y)z + h(x,y).$$

Assume that $k[x,y,z]/(f) \cong k^{[2]}$. Then f is a coordinate over $k[x]$.

In fact, the proof of this theorem shows that, under the asssumption that $k[x,y,z]/(f) \cong k^{[2]}$, the polynomial $g(x,y)$ is in fact a polynomial in a coordinate, i.e. $g(x,y) = \tilde{g}(c(x,y))$ for some $\tilde{g}(t) \in k[t]$ and some coordinate $c(x,y) \in k[x,y]$. This suggests the following generalization of this theorem.

Theorem 7.2. *Let R be an arbitary $\mathbb{Q}$-algebra and let $f(x,y,z) \in R[x,y,z]$ be a polynomial of the form*

$$f(x,y,z) = g(c(x,y))z + h(x,y),$$

for some coordinate $c(x,y) \in R[x,y]$ and some polynomial $g(t) \in R[t]$ and $h(x,y) \in R[x,y]$. Assume that $R[x,y,z]/(f) \cong R^{[2]}$. Then f is a coordinate over $R[x]$.

Proof. It is possible to assume, without loss of generality, that $c(x,y) = x$. So $f = g(x)z + h(x,y)$.

Now let $\mathfrak{q} \subseteq R[x]$ be a prime ideal and take $\mathfrak{p} := \mathfrak{q} \cap R$. Since $R[x,y,z]/(f) \cong R^{[2]}$, also $k_\mathfrak{p}[x,y,z]/(f_\mathfrak{p}) \cong k_\mathfrak{p}^{[2]}$. By Theorem 7.1, $f_\mathfrak{p}$ is a coordinate over $k_\mathfrak{p}[x]$ in $k_\mathfrak{p}[x,y,z]$. Let $g \in k[x,y,z]$ be a polynomial such that $(f_\mathfrak{p}, g_\mathfrak{p}) \in \mathrm{Aut}_{k_\mathfrak{p}[x]}(k_\mathfrak{p}[x,y,z])$ and let φ be the natural map from the polynomial ring $k_\mathfrak{p}[x,y,z]$ to the polynomial ring $(R[x]_\mathfrak{q}/\mathfrak{q}R[x]_\mathfrak{q})[y,z]$. Then

$$(f_\mathfrak{q}, g_\mathfrak{q}) = (\varphi(f_\mathfrak{p}), \varphi(g_\mathfrak{p})) \in \mathrm{Aut}_{R[x]_\mathfrak{q}/\mathfrak{q}R[x]_\mathfrak{q}}(R[x]_\mathfrak{q}/\mathfrak{q}R[x]_\mathfrak{q})[y,z],$$

so $f_\mathfrak{q}$ is a coordinate over the residue field $R[x]_\mathfrak{q}/\mathfrak{q}R[x]_\mathfrak{q}$.

Hence, applying Theorem 3.4 to the ring $R[x]$ and the polynomial f, it follows that f is a coordinate over $R[x]$. $\square$

The following example shows that it is in general not true that a polynomial that is residually a polynomial in a coordinate, is itself a polynomial in a coordinate.

Example 7.3. Take $R := \mathbb{C}[t]$ and $f := x^2 + ty \in R[x,y]$. Then $f_\mathfrak{p}$ is a coordinate in $k_\mathfrak{p}[x,y]$ for every prime ideal $\mathfrak{p} \neq 0$ and $f_{(0)} = x^2$. The polynomial f itself is not a polynomial in a coordinate, though.

Finally, this section exhibits several examples showing that most results from the previous sections do not hold for more than two variables.

This first example shows that Theorem 3.4 cannot be extended to higher dimensions.

Example 7.4. Let $R := \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ and $f := \bar{x}X_1 + \bar{y}X_2 + \bar{z}X_3 \in R[X_1,X_2,X_3]$. Since $(\bar{x},\bar{y},\bar{z}) = (1)$, it follows that for every prime ideal $\mathfrak{p}$ of R either $\bar{x}$, $\bar{y}$, or $\bar{z}$ does not belong to $\mathfrak{p}$, which implies that $f_\mathfrak{p}$ is a (linear) coordinate in $R_\mathfrak{p}[X_1,X_2,X_3]$. However, it is shown in [ER01] that f is not a coordinate in $R[X_1,X_2,X_3]$. So the implication (2) $\Rightarrow$ (1) of Theorem 3.4 does not hold.

Concerning a possible generalization of Threom 3.5: here almost nothing is known. Even in the case $n = 3$ and $R = \mathbb{C}$ it is still an open problem if the assumption that $\mathbb{C}[X_1,X_2,X_3]/(f) \cong_\mathbb{C} \mathbb{C}^{[2]}$ implies that f is a coordinate in $\mathbb{C}[X_1,X_2,X_3]$. This is the 3-dimensional version of the well-known Abhyankar-Sathaye Conjecture. Some special results are obtained in [Sat76], [Rus76], and [BD94]. See also Chapter 5, Section 3 of [Ess00].

The next example shows that also Theorem 4.2 cannot be extended to higher dimensions.

Example 7.5. Let R and f be as in Example 7.4. Then, as remarked above, f is not a coordinate in $R[X_1,X_2,X_3]$. However, regarded as an element of $R[X_1,X_2,X_3,X_4]$ the polynomial f is a coordinate: one can easily verify that $(f, X_1+\bar{x}X_4, X_2+\bar{y}X_4, X_3+\bar{z}X_4)$ is an R-automorphism of $R[X_1,X_2,X_3,X_4]$.

Concerning Theorem 5.2, it was already shown in [MYZ97] that a similar result does not hold in dimensions greater than two.

Example 7.6. Take $R := \mathbb{R}$ and take $\varphi \in \mathrm{End}_\mathbb{R}(\mathbb{R}[X_1,X_2,X_3])$ defined by $\varphi := (X_1 + X_2X_3, X_2 - X_1X_3, X_3)$. Then $\det(J\varphi) = 1 + X_3^2$, so φ is not an $\mathbb{R}$-automorphism of $\mathbb{R}[X_1,X_2,X_3]$. On the other hand, it was shown in [MYZ97] that φ does send linear coordinates in $\mathbb{R}[X_1,X_2,X_3]$ to coordinates in $\mathbb{R}[X_1,X_2,X_3]$.

Concerning a possible extension of the first part of Theorem 6.4 to dimension greater than two: in [Sha92], Shastri obtained the following embedding of $\mathbb{R}$ in $\mathbb{R}^3$:

$$\alpha := (t^3 - 3t, t^4 - 4t^2, t^5 - 10t).$$

This is a polynomial representation of the trefoil knot and hence it is not rectifiable over $\mathbb{R}$. So even for the field case an extension does not hold in dimension three. On the other hand, Sastri's conjecture that $\alpha\colon \mathbb{C} \to \mathbb{C}^3$ is not rectifiable over $\mathbb{C}$ is still an open problem. So in dimension three for the case $R = \mathbb{C}$ it is still unknown if an extension of Theorem 6.4 holds.

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Department of Mathematics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

E-mail address: `essen@math.kun.nl`

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003

E-mail address: `petervr@nmsu.edu`